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NUMERICAL MODEL OF NONSTATIONARY FLOW OF A VISCOPLASTIC FLUID IN A ROTATIONAL VISCOSIMETER

V. P. Beskachko,[a] **O. A. Golovnya,**[a] **and A. E. Korenchenko**[b]

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The results of numerical solution of the problem on flow of an incompressible viscoplastic fluid in the gap between two rotating cylinders have been presented. The criterion of formation of the boundaries of stagnant zones has been developed. A comparison of the results of numerical calculations to the analytical solution has been made. The time of establishment of stationary flow in the gap has been determined.

We know of the analytical solutions for steady-state flows of viscoplastic materials in units with a simple geometry. In particular, the problem on one-dimensional stationary flow of a fluid with a linear [1] and nonlinear [2] viscoplasticity in the gap between two coaxial cylinders has been solved; flow between two concentric rotating spheres has been considered in [3], whereas flow between two coaxial cones has been the focus of [4]. In investigating flow of a viscoplastic fluid in cases of complex geometry and to solve nonstationary problems one uses different numerical models. Among them is, e.g., the double-viscosity model [5]. However, in these models, the equations of motion of the fluid throughout the region are solved, but the yield stress is disregarded and the zones of solid-state motion (stagnant zones) are not considered. In the present work, we have numerically analyzed the establishment of flow of a linear viscoplastic fluid in the gap between two cylinders with allowance for the yield stress. We have considered the processes of formation and evolution of stagnant zones and have made a comparison to the existing analytical solution [1].

Formulation of the Problem. In a cylindrical coordinate system, we consider the motion of a viscoplastic medium in the gap between two cylinders. The external cylinder begins rotation at the instant $t = 0$ with a constant angular velocity ω, whereas the internal cylinder is fixed. In the case where the length of the cylinders is much larger than their radius, only one component of the fluid velocity — V_φ — is nonzero in the approximation of the axial flow symmetry. The equation of motion of the viscoplastic medium will be written as

$$\frac{\partial V_\varphi}{\partial t} = -\frac{1}{\rho}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\varphi})\right). \tag{1}$$

The governing relations for a linear viscoplastic material have the form [6]

$$\tau_{ij} = -\left(\eta + \frac{\tau_0}{\sqrt{(\dot{\varepsilon}, \dot{\varepsilon})/2}}\right)\dot{\varepsilon}_{ij} \quad \text{for} \quad \sqrt{(\tau, \tau)/2} > \tau_0\,, \quad \dot{\varepsilon}_{ij} = 0 \quad \text{for} \quad \sqrt{(\tau, \tau)/2} \le \tau_0\,. \tag{2}$$

The only nonzero component $\dot{\varepsilon}$ in the adopted approximation of long cylinders and axisymmetric flow is $\dot{\varepsilon}_{r\varphi} = \partial V_\varphi/\partial r - V_\varphi/r$.

Equation (1) is true just in the region of viscous flow; the remaining part of the fluid represents a stagnant zone and moves as a solid body. Adhesion conditions are fulfilled at the solid boundaries. The fluid together with the

[a]South-Ural State University, 76 Lenin Ave., Chelyabinsk, 454080, Russia; [b]Institute of Metallurgy, Ural Branch of the Russian Academy of Sciences, 101 Amundsen Str., Ekaterinburg, 620016, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 80, No. 1, pp. 12–14, January–February, 2007. Original article submitted July 4, 2005; revision submitted October 14, 2005.

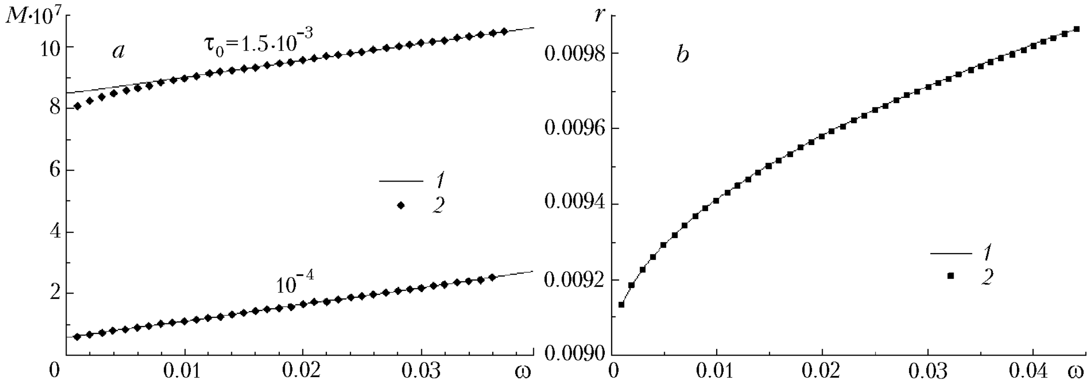

Fig. 1. Applied rotational moment (a) and radius of the stagnant zone (b) vs. angular rotational velocity of the external cylinder: 1) analytical calculation; 2) numerical calculation. M, N·m; r, m; ω, $\sec^{-1}$.

cylinders is at rest at the initial instant of time; at the instant $t = 0$, the external cylinder acquires a velocity ω with a jerk.

Numerical Method of Solution. To solve problem (1) and (2) we used a difference scheme with a uniform grid containing N nodes. The space derivatives were approximated accurate to $(\Delta x)^2$ according to the central-difference scheme; the time derivative was determined in advance in time accurate to Δt. The system of difference equations was solved by the marching method [7]. The position of stagnant zones at the instant t^{n+1} was taken from the previous step.

Conditions (2) imply that for the viscous flow to exist in the vicinity of a certain point of the region, the requirement $\sqrt{(\dot{\varepsilon}, \dot{\varepsilon})/2} > 0$ must be met, whereas in the region of solid-state motion we have $\sqrt{(\dot{\varepsilon}, \dot{\varepsilon})/2} \equiv 0$. The boundary separating the regions of viscous flow and the stagnant zones is a priori unknown and varies with time until the stationary state is reached. The difficulty of numerical description of nonstationary flows of a viscoplastic fluid is that the viscous-flow regions should be determined at each time step using conditions (2), which can be fulfilled in the difference interpretation of the problem only approximately. In the work, we used the following criterion for determination of the boundary of a stagnant zone: flow is assumed to be viscous for $\sqrt{(\dot{\varepsilon}, \dot{\varepsilon})/2} \geq \delta$, and the condition $\sqrt{(\dot{\varepsilon}, \dot{\varepsilon})/2} < \delta$ is fulfilled in the region of solid-state motion. As the analysis has shown, the selection of $\delta = \tau_0 R_{ex}^2/(N\nu^2\rho)$ yields a sufficient accuracy when the analytical and numerical dependences are compared.

Discussion of the Results. At the instant of the beginning of motion, the strain rate is nonzero only in the region immediately adjacent to the external cylinder; the entire remaining fluid is at rest. The stagnant zone adjacent to the internal cylinder is very rapidly dissipated. After a time, the value of the modulus of the strain-rate tensor at the points near the external cylinder can be less than δ, which points to the formation of a stagnant zone adherent to the external cylinder.

Numerical calculations were carried out with the following conditions: radius of the external cylinder R_{ex} = 0.01 m, radius of the internal cylinder R_{in} = 0.009 m, angular velocity of the external cylinder ω 10^{-3} to 0.5 $\sec^{-1}$, and yield stress in the range 10^{-4} Pa $\leq \tau_0 \leq 2{\cdot}10^{-3}$ Pa.

As has been indicated above, the "dissipation" of the internal stagnant zone occurs in the initial step of motion of the external cylinder; somewhat later, the formation of a stagnant zone "adherent" to the external cylinder begins. Then the thickness of this zone increases during a certain period followed by the regime of stationary rotation. Once the stationary state has been reached, we compute the radius of the stationary stagnant zone (as long as it is formed) and the moment M of viscous forces acting on the external cylinder (this moment is equal to the rotational moment necessary for maintaining motion with a prescribed angular velocity). Figure 1a shows the analytical and numerical dependences $M = M(\omega)$ for fluids with different values of the yield stress. As the calculations show, no stagnant zone is formed for the fluid with $\tau_0 = 10^{-4}$ Pa in the range of angular velocities considered, the character of the dependence $M(\omega)$ is linear, and the difference between the analytical and numerical results is no higher than 0.1%. The angle of inclination of this straight line is used in rheological practice to measure the viscosity η, whereas the segment intercepted by it on the ordinate axis is used to measure the yield stress. For the fluid with $\tau_0 = 1.5{\cdot}10^{-3}$ Pa,

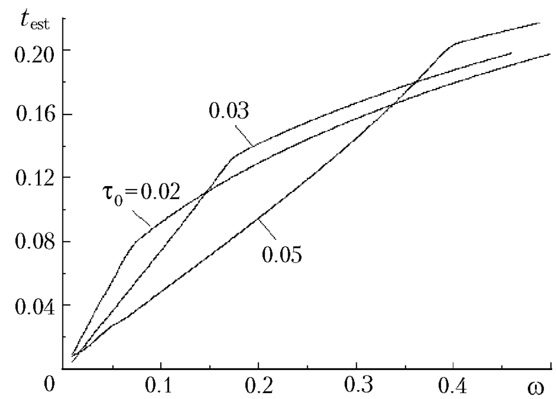

Fig. 2. Times of establishment of flow in the viscosimeter vs. angular velocity and yield stress. t_{est}, sec; ω, sec^{-1}.

the stagnant zone is formed for the angular velocities $\omega < 0.01$ sec^{-1}. This is reflected on the character of the rotational moment plotted as a function of the angular velocity. As is clear from the figure, the numerical curve falls off from the analytical linear dependence obtained under the assumption that no stagnant zone is formed in the fluid and extended to the region of angular velocities where this condition is not fulfilled. For the range of angular velocities for which no stagnant zone is formed, the analytical and numerical dependences are coincident.

Figure 1b shows the numerical and analytical dependences of the value of the stagnant-zone radius on the angular velocity of the external cylinder. As is clear from the figure, the differences in the results are very small, which points to the applicability of the approximate criterion proposed above to formation of the boundaries of stagnant zones. Using the data of the numerical calculation, we can determine the times of establishment of flows in the rotational viscosimeter under the physical assumptions made. Figure 2 shows the establishment time as a function of the rotational velocity of the external cylinder for different values of the yield stress. As is seen in the figure, each plot represents a curve with a pronounced derivative jump. The left-hand part of the curve corresponds to the conditions under which the formation of a stagnant zone occurs, whereas the right-hand part corresponds to the absence of this zone. The stationary state is established over a period shorter than 300 msec, which is much less than the period of rotation for the angular velocities indicated in the figure. Since the stationary rotation is reached in very small intervals, the establishment of the stationary state in the rotational viscosimeter is mainly determined by the stability of the mechanical system setting the cylinder in rotation.

Thus, in the present work, we have obtained a finite-difference description of the nonstationary process of establishment of flow of a viscoplastic fluid and formation of a stagnant zone. By comparing the results of numerical calculations to the existing analytical solution, we have developed a criterion for formation of the regions of viscous and solid-state motions. It can be used in numerical solution of problems on nonstationary flows of a viscoplastic fluid under conditions of a more complex geometry. We have evaluated the time of establishment of stationary flow in the gap between cylinders.

NOTATION

M, moment of viscous forces acting on the external cylinder, N·m; N, number of spatial nodes in the difference scheme; R_{in} and R_{ex}, radii of the internal and external cylinders of the viscosimeter, m; t, time; t_{est}, time of establishment of stationary flow between the cylinders, sec; t^n, time of the difference scheme, nth step, sec; V_φ, azimuthal component of the fluid velocity, m/sec; z, r, φ, cylindrical coordinates, m, m, rad; Δt, time step of the difference scheme, sec; Δx, space step of the difference scheme, m; δ, dimensionless numerical criterion for determination of the boundary motion of a solid body–viscous flow; $\dot{\varepsilon}$ and $\dot{\varepsilon}_{ij}$, strain-rate tensor and its components, sec^{-1}; η, coefficient of dynamic viscosity, Pa·sec; ν, coefficient of kinematic viscosity, m^2/sec; ρ, density of the fluid, kg/m^3; τ and τ_{ij}, stress tensor and its components, Pa; τ_0, yield stress of the fluid, Pa; ω, angular velocity of the external cylinder, sec^{-1}. Subscripts: ex, external; in, internal; 1 and 2, No. of time step; est, establishment.

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